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REDUCIBILITY OF SOME INDUCED REPRESENTATIONS OF p-ADIC UNITARY GROUPS

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Abstract. In this paper we study reducibility of those representations of quasi-split unitary p-adic groups which are parabolically induced from supercuspidal representations of general linear groups. For a supercuspidal representation associated via Howe's construction to an admissible character, we show that in many cases a criterion of Goldberg for reducibility of the induced representation reduces to a simple condition on the admissible character.

1. Introduction

Let K be a quadratic extension of a p-adic field F of characteristic zero and odd residue characteristic. Let G' and G'' be the F-rational points of the quasi-split unitary groups in $2n$ and $2n+1$ variables, respectively, defined with respect to the extension K/F. Let $G = GL_n(K)$. Denote the kernel of the norm map from $K^\times$ to $F^\times$ by K^1. The group G', resp. G'', has a maximal parabolic subgroup P', resp. P'', with Levi factor isomorphic to G, resp. $G \times K^1$. Let π be an irreducible unitary supercuspidal representation of G, and ξ a character of K^1. Define a supercuspidal representation of Π_ξ of $G \times K^1$ by $\Pi_\xi(x,\alpha) = \pi(x)\,\xi(\det_0(x\eta(x))\alpha)$, for x in G and $\alpha \in K^1$. Here $\det_0$ is the determinant on G, and η is the automorphism of G taking x to ${}^t\bar{x}^{-1}$, where the bar denotes the usual action of the non-trivial element of $Gal(K/F)$ on matrices with entries in K. Set

$$I(\pi) = \mathrm{Ind}_{P'}^{G'}(\pi \otimes 1)$$

and

$$I(\Pi_\xi) = \mathrm{Ind}_{P''}^{G''}(\Pi_\xi \otimes 1).$$

As it is a necessary condition for reducibilty of $I(\pi)$, and also for $I(\Pi_\xi)$, we assume that π is equivalent to $\pi \circ \eta$. In [G2], Goldberg proves that, under this assumption, $I(\pi)$ is reducible, resp. $I(\Pi_\xi)$ is irreducible, if and only if the sum of two particular η-twisted orbital integrals vanishes for every choice of matrix coefficient of π.

Suppose that π arises via the construction of Howe ([H]) from an admissible character θ of the multiplicative group of a tamely ramified degree n extension E of K. We show that π is equivalent to $\pi \circ \eta$ if and only $\theta \circ \sigma = \theta^{-1}$ for some involutive automorphism of E/F which is non-trivial on K. In this paper, we prove that, for many such π, Goldberg's reducibility criterion reduces to a simple condition on θ. If L is the fixed field of σ, then either $\theta \,|\, L^\times$ is trivial or is equal to the quadratic

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character of $L^\times$ associated to E/L by class field theory. When E is ramified over L and $\theta\,|\,L^\times$ is trivial, we show that the sum of η-twisted orbital integrals which appears in the reducibility criterion is non-zero for a particular choice of matrix coefficient of π. When E is unramified over L, we get a similar result under some additional assumptions on θ. In an earlier paper ([MR]), using a reducibility criterion of Shahidi ([Sh]), we obtained the same type of results for representations of split classical groups induced from self-contragredient supercuspidal representations of general linear groups. Many of the results of this paper are proved by modifying proofs of analogous results of [MR].

In §2, we derive the relation between the equivalence of π and $\pi\circ\eta$ and existence of σ as above. In particular, it follows from a result of Adler ([A]) that existence of such an involution σ guarantees existence of such supercuspidal representations π. We also discuss properties of the Howe factorization of θ relative to σ.

The η-twisted orbital integrals in Goldberg's criterion can be expressed as integrals over certain sets of fixed points in G of an involutive anti-automorphism φ of $\mathfrak{gl}_n(K)$. The third section contains a description of the action of φ on filtrations of the parahoric subalgebra attached to the extension E/K, and on related subgroups of G.

The representation π is induced from an irreducible representation κ of an open compact subgroup H_0 of G. In §4, we state the reducibility criterion of [G2], and show that for an appropriately chosen finite sum f_π of matrix coefficients of π, each of the two relevant η-twisted orbital integrals $\Phi_\eta(h_k, f_\pi)$, $k=1,2$, reduces to the integral of the character of κ over a certain φ-invariant subset of H_0.

In §5, we give some values of the character of κ, and summarize some results from [MR] relating properties of κ and certain extensions of F contained in E. We prove that if κ is one-dimensional, then $\Phi_\eta(h_k, f_\pi) > 0$, $k=1,2$.

Up to a character of H_0, the inducing representation κ is a tensor product of finitely many representations κ_i corresponding to the Howe factors θ_i, $i=1,\dots,r$, of the admissible character θ. In §6, we show that if a Heisenberg representation is used in the construction of one of these factors, then the character χ_i of κ_i is real-valued on the set of φ-invariant points in H_0. We then compute the value of certain signs appearing in the formula for χ_i.

Next, in §7, we consider the case when the representation κ_r is defined in terms of a cuspidal representation of a finite general linear group. Assuming that κ_i is one-dimensional for $1\le i\le r-1$, we outline how to modify the arguments of [MR] to express $\Phi_\eta(h_k, f_\pi)$, $k=1,2$, in terms of values of θ and sums of χ_r over various subsets of H_0. As shown in [MR], these sums of values of χ_r can be expressed in terms of Deligne-Lusztig characters of non-connected finite reductive groups which were computed in [MR]. This allows us to relate the signs of $\Phi_\eta(h_k, f_\pi)$, $k=1,2$, and $\theta\,|\,L^\times$.

The main results of the paper are Theorems 8.1 and 8.3. We state conditions on $\theta\,|\,L^\times$ which guarantee that $\Phi_\eta(h_k, f_\pi) > 0$, $k=1,2$, and hence that $I(\pi)$ is irreducible, resp. $I(\Pi_\xi)$ is reducible.

In analogy with the situation in [Sh], the reducibility criterion of [G2] can be interpreted in terms of the conjectural theory of twisted endoscopy ([KS1],[KS2]). For $n=2$ and 3, this is discussed in [G1] and in §4 of [G2], respectively. Under the conditions on θ given in §8 of this paper, the representation π should be a lift from the unitary group in n variables (see §§4,6 of [G2]).

2. Howe factorizations of admissible characters

Let F be a p-adic field of characteristic zero and odd residual characteristic. If F' is a finite extension of F, we will use the notation $\mathcal{O}_{F'}$, $\mathfrak{p}_{F'}$, and $\varpi_{F'}$ for the ring of integers in F', maximal ideal in the ring of integers, and a uniformizer in F', respectively. The norm and trace maps from F' to F will be denoted by $N_{F'/F}$ and $\mathrm{tr}_{F'/F}$, respectively. Fix a quadratic extension K of F. For $n \geq 2$, set $G = GL_n(K)$; we let $x \mapsto \bar{x}$ denote the action of the non-trivial element of the Galois group of K/F on G (apply the automorphism to matrix entries). Set $\eta(x) = {}^t\bar{x}^{-1}$. Let π be an irreducible supercuspidal representation of G such that $\pi \circ \eta$ is equivalent to π (denoted by $\pi \circ \eta \sim \pi$). Now suppose that π arises via Howe's construction from an admissible character θ of $E^\times$, where E/K is tamely ramified of degree n. Note that E/F may not be Galois; we use the notation $Aut(E/F)$ to refer to the set of automorphisms of E that fix F pointwise, and similarly for $Aut(E/K)$. Note that θ is admissible over K, but might not be admissible over F. Assume that π (hence θ) is unitary. The above condition on π translates into a condition on θ.

Lemma 2.1. *$\pi \sim \pi \circ \eta$ if and only if there exists an involution $\sigma \in Aut(E/F)$ such that $\sigma \,|\, K \not\equiv id$ and $\theta \circ \sigma = \theta^{-1}$.*

Proof. ($\Rightarrow$) Take an embedding τ of E into the algebraic closure of F having the property that $\tau \,|\, K$ is the non-trivial element of $Gal(K/F)$. Let $E' = \tau(E)$. Then we can set $\theta'(\tau(\alpha)) = \theta(\alpha)$, $\alpha \in E^\times$ and observe that θ' is attached to the representation $x \mapsto \pi(\bar{x})$. But we also know that $x \mapsto \pi({}^t x^{-1})$ is attached to θ^{-1}. So the condition on π forces θ' and θ^{-1} to be conjugate (over K): there is a field isomorphism $\tau' : E' \to E$ which fixes K pointwise such that $\theta^{-1}(\tau'(\alpha')) = \theta'(\alpha')$, $\alpha' \in E'^\times$. Set $\sigma = \tau' \circ \tau$. Then $\sigma \in Aut(E/F)$. The automorphism σ has the property that $\sigma \,|\, K$ is the non-trivial element of $Gal(K/F)$ and also that $\theta \circ \sigma = \theta^{-1}$.

What remains is to show that σ is an involution. Note that $\theta \circ \sigma^2 = \theta$, and $\sigma^2 \in Aut(E/K)$. Suppose the order of σ^2 is $k > 1$. Write E^{σ^2} for the fixed field of σ^2. Then $[E : E^{\sigma^2}] \leq k$. But $1, \sigma^2, \sigma^4, \ldots, \sigma^{2(k-1)}$ are k distinct automorphisms of E fixing E^{σ^2} pointwise. This shows that E/E^{σ^2} is normal, with $[E : E^{\sigma^2}] = k$, and therefore Galois. Since $\theta \circ \sigma^2 = \theta$, we find that for any $t \in E^\times$, $\theta\big(\frac{t}{\sigma^2(t)}\big) = 1$. By Hilbert 90, this shows that θ is trivial on the elements of norm 1, so θ factors through the norm $N_{E/E^{\sigma^2}}$. This contradicts the admissibility of θ, proving that σ is indeed an involution.

($\Leftarrow$) If there is an involution σ as in the statement of the lemma, then, as above, $x \mapsto \pi(\bar{x})$ is equivalent to $x \mapsto \pi({}^t x^{-1})$, so $\pi \sim \pi \circ \eta$. □

Note that in contrast to the situation in [MR], σ acts non-trivially on the base field K over which the supercuspidal representation is defined.

Lemma 2.2. *Suppose E/K is a tamely ramified extension of degree n. The following are equivalent:*
(i) There exists an involution $\sigma \in Aut(E/F)$ such that $\sigma\big|_K \not\equiv id$.
(ii) There exist irreducible unitary supercuspidal representations π of G associated by the construction of Howe to admissible characters θ of $E^\times$ and satisfying $\pi \sim \pi \circ \eta$.

Proof. Part (ii) implies (i) by Lemma 2.1.

$(i) \Rightarrow (ii)$: The fixed field of σ is of index 2 in E. The argument given in the proof of Theorem 6.1 of [A] shows that there exists a character θ of $E^\times$ that is admissible over F and such that $\theta \circ \sigma = \theta^{-1}$. Admissibility over F implies admissibility over K, and (ii) follows by Lemma 2.1. □

Assume that π and θ are as in Lemma 2.1. The admissible character θ of $E^\times$ has a Howe factorization (see [H], [M]):

$$\theta = (\Lambda \circ N_{E/K})\theta_r(\theta_{r-1} \circ N_{E/E_{r-1}}) \cdots (\theta_2 \circ N_{E/E_2})(\theta_1 \circ N_{E/E_1}).$$

Here θ uniquely determines the tower of fields $K = E_0 \subset E_1 \subset \cdots \subset E_r = E$ and Λ, $\theta_1, \dots, \theta_r$ are quasi-characters of $E_0^\times$, $E_1^\times, \dots, E_r^\times$, respectively. Comparison of the Howe factorizations of θ and $\theta \circ \sigma$ shows that $\sigma(E_i) = E_i$ for each i, although we shall see that σ does not fix E_i pointwise. Each quasi-character θ_i is generic over E_{i-1} ([H]). The conductoral exponents are unique and satisfy

$$f_E(\theta_1 \circ N_{E/E_1}) > \cdots > f_E(\theta_r) > 0.$$

If $f_E(\Lambda \circ N_{E/K}) \le f_E(\theta_1 \circ N_{E/E_1})$, note that it is possible to absorb $\Lambda \circ N_{E/K}$ into $\theta_1 \circ N_{E/E_1}$ and write $(\Lambda \circ N_{E/K})(\theta_1 \circ N_{E/E_1}) = (\theta_1' \circ N_{E/E_1})$ for a θ_1' that is still generic over E_1. Because of this, we can choose θ_1 such that either $\Lambda \equiv 1$ or $f_E(\Lambda \circ N_{E/K}) > f_E(\theta_1 \circ N_{E/E_1})$. For each $i = 1, \dots, r-1$, choose an element $c_i \in E_i$ that "represents" θ_i in the sense that

$$\theta_i(1+x) = \psi\big(\mathrm{tr}_{E_i/K}(c_i x)\big), \qquad \text{for} \quad x \in \mathfrak{p}_{E_i}^{\left[\frac{f_{E_i}(\theta_i)+1}{2}\right]},$$

where $\psi = \psi_0 \circ \mathrm{tr}_{K/F}$ and ψ_0 is a character of the additive group F with conductor $\mathfrak{p}_F$; we must have $c_i \in \mathfrak{p}_{E_i}^{-f_{E_i}(\theta_i)+1} \setminus \mathfrak{p}_{E_i}^{-f_{E_i}(\theta_i)+2}$ (see [H], [M]). Note that the genericity of θ_i implies that c_i generates E_i over E_{i-1}. If $i = r$ and $f_E(\theta_r) > 1$, choose c_r as above.

Let σ be as in Lemma 2.1.

Lemma 2.3. *The characters Λ and θ_i, and the elements c_i can be chosen so that*

(i) *Λ, θ_i are unitary,*
(ii) *$\Lambda \circ N_{E/K} \circ \sigma = (\Lambda \circ N_{E/K})^{-1}$, $\qquad \theta_i \circ N_{E/E_i} \circ \sigma = (\theta_i \circ N_{E/E_i})^{-1}$,*
(iii) *$\sigma(c_i) = -c_i$, if $f_E(\theta_i) > 1$.*

Proof. The proof is the same as the proof of Lemma 2.5 in [MR], noting that the adjustments made in that proof to the various characters do not affect whether or not $f_E(\Lambda \circ N_{E/K}) > f_E(\theta_1 \circ N_{E/E_1})$ (and hence whether or not $\Lambda \equiv 1$). □

From now on we assume that Λ, θ_i and c_i are as in Lemma 2.3.

3. Filtrations and the map φ

Let the notation be as in §2. We will define an antimorphism φ of $\mathfrak{gl}_n(K)$ whose action on E is given by σ, and so that the integrals we will be discussing can be expressed in terms of integrals over certain sets of φ-invariant points in a subgroup H_0. The subgroup H_0 is the intersection with G of the subgroup H of $GL_{2n}(F)$ defined in [MR], and the map φ is the restriction to G of the map φ defined there, so various properties of these maps relative to intermediate extensions, filtrations, and parahoric subgroups will follow immediately from results of [MR].

Let L be the fixed field of σ in E. We begin by fixing embeddings $L \hookrightarrow \mathfrak{gl}_n(F)$ and $E \hookrightarrow \mathfrak{gl}_2(L) \subset \mathfrak{gl}_{2n}(F)$ and a symmetric matrix $s \in GL_n(F)$ such that $w =$

$\begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}$ satisfies $w^{-1}\,{}^t\gamma w = \sigma(\gamma)$ for every $\gamma \in E \subset \mathfrak{gl}_{2n}(F)$. Then, as in [MR], we define the map $\varphi : \mathfrak{gl}_{2n}(F) \to \mathfrak{gl}_{2n}(F)$ by

$$\varphi(X) = w^{-1}\,{}^tXw.$$

By Lemma 3.4 of [MR], there is a symmetric matrix $\mathcal{S} \in GL_n(F) \subset GL_n(K)$ such that for $X \in \mathfrak{gl}_n(K)$, we have

$$\varphi(X) = \mathcal{S}^{-1}\,{}^t\overline{X}\mathcal{S}, \tag{3.1}$$

where here and from now on ${}^t\cdot$ refers to the transpose in $\mathfrak{gl}_n(K)$ and $\overline{X}$ refers to the conjugate of X by σ acting on the entries of X. If E/L is ramified, take a_L to be a non-square root of unity in L. Otherwise, let $a_L = \varpi_L$. Then let

$$h_1 = \mathcal{S}^{-1} \qquad \text{and} \qquad h_2 = a_L h_1 = a_L \mathcal{S}^{-1}.$$

Note that h_1 and h_2 are hermitian as elements of $GL_n(K)$ relative to the action of σ described above. Because of the choice of a_L, $\det(h_1)$ and $\det(h_2) = N_{E/K}(a_L)\det(h_1)$ both belong to $F^\times$ and, under the assumptions on E and σ (see Lemma 2.2(i)) they lie in different cosets of $N_{K/F}(K^\times)$. This implies that h_1 and h_2 are representatives of the two equivalence classes of hermitian matrices in G.

We define various subalgebras and subgroups as in [MR]. The parahoric $\mathcal{O}_F$-subalgebra $\mathcal{B} \subset \mathfrak{gl}_{2n}(F)$ attached to the embedding $E \hookrightarrow \mathfrak{gl}_{2n}(F)$ is defined by

$$\mathcal{B} = \{X \in \mathfrak{gl}_{2n}(F) \mid X\mathfrak{p}_E^k \subset \mathfrak{p}_E^k, \text{for all } k\}.$$

For any integer j, we also define

$$\mathcal{B}_j = \{X \in \mathfrak{gl}_{2n}(F) \mid X\mathfrak{p}_E^k \subset \mathfrak{p}_E^{k+j}, \text{for all } k\}.$$

The parahoric subgroup $P \subset GL_{2n}(F)$ is the units

$$P = \mathcal{B}^\times,$$

and we let

$$P_0 = P; \qquad P_j = 1 + \mathcal{B}_j, \qquad \text{for} \quad j \geq 1.$$

We define a function ν on $\mathfrak{gl}_{2n}(F)$ by $\nu(X) = j$, where j is the unique integer such that $X \in \mathcal{B}_j \setminus \mathcal{B}_{j+1}$. Note that if $X \in E$, then $\nu(X) = \mathrm{ord}_E(X)$. We embed $\mathfrak{gl}_{[E:E_i]}(E_i)$ in $\mathfrak{gl}_n(E_0) = \mathfrak{gl}_n(K) \subset \mathfrak{gl}_{2n}(F)$ as the set of all elements of $\mathfrak{gl}_n(K)$ that centralize $E_i \subset E \subset \mathfrak{gl}_n(K)$. We will refer to this realization of $\mathfrak{gl}_{[E:E_i]}(E_i)$ as M_i. In this situation, for $i = 0, \ldots, r$, we will define

$$\begin{aligned} \mathcal{B}_j(i) &= \{X \in M_i \mid X\mathfrak{p}_{E_i}^k \subset \mathfrak{p}_{E_i}^{k+j}, \text{for all } k\} = \mathcal{B}_j \cap M_i, \\ P_j(i) &= P_j \cap M_i, \end{aligned}$$

and

$$\mathcal{B}(i) = \mathcal{B}_0(i), \qquad P(i) = P_0(i) = \mathcal{B}(i) \cap P.$$

The only difference from the definitions of [MR] is that here $\mathcal{B}_j(0) = \mathcal{B}_j \cap \mathfrak{gl}_n(K) \subsetneq \mathcal{B}_j$ (since $E_0 = K$), while in the previous paper $E_0 = F$ and $\mathcal{B}_j(0) = \mathcal{B}_j$.

Lemma 3.1. *(*[MR]*, Corollary 3.5). For* $0 \leq i \leq r$,
(i) $\varphi(M_i) = M_i$,
(ii) $\varphi(\mathcal{B}_j(i)) = \mathcal{B}_j(i)$, $j \in \mathbb{Z}$,
(iii) $\varphi(P_j(i)) = P_j(i)$, $j \geq 0$. □

For $1 \le i \le r$, write $\ell_i = \left[\frac{f_E(\theta_i \circ N_{E/E_i})}{2}\right]$. Set

$$\begin{aligned} H_0 &= E^\times P_{\ell_r}(r-1)\cdots P_{\ell_2}(1)P_{\ell_1}(0), \\ \mathcal{K}_i &= P_{\ell_r}(r-1)\cdots P_{\ell_{i+1}}(i), \quad 0 \le i \le r-1; \qquad \mathcal{K}_r = \{1\}, \\ \mathcal{L}_i &= P_{\ell_i}(i-1)\cdots P_{\ell_1}(0), \quad 1 \le i \le r. \end{aligned}$$

If H, K_i, L_i are the corresponding subgroups defined in §3 of [MR], then we note that $H_0 = H \cap G$, $\mathcal{K}_i = K_i \cap G$, $\mathcal{L}_i = L_i \cap G$. For any subset $A \subset \mathfrak{gl}_n(K)$, we will write A^φ for the φ-fixed points in A.

Lemma 3.2. *(i) ([MR], Corollary 3.8). Let $x \in H_0^\varphi$, and $1 \le i \le r$. Then there exist $y \in (E^\times \mathcal{K}_i)^\varphi$ and $z \in \mathcal{L}_i$ such that $x = yz$.*
(ii) ([MR], Lemma 3.9). Let $0 \le i \le r$, $j \ge 1$, and $\tau \in (H_0 \cap M_i)^\varphi$. Then the map $x \mapsto x\tau\varphi(x)$ from $P_j(i)$ to $\big(\tau P_j(i)\big)^\varphi$ is onto. □

4. Goldberg's reducibility criterion

Suppose that $\widetilde{\omega}$ is a character of $K^\times$ of the form $\widetilde{\omega}(z) = \omega(z/\bar{z})$, $z \in K^\times$, for some character ω of the kernel K^1 of $N_{K/F}$. Let $C(G, \widetilde{\omega})$ be the space of locally constant complex-valued functions on G which are compactly supported modulo the centre Z_K of G, and satisfy $f(zg) = \widetilde{\omega}^{-1}(z)\, f(g)$, $z \in Z_K$, $g \in G$. Let Z_F denote the F-scalar matrices in G. Given $x \in G$, let

$$G_{x\eta, Z_F} = \{\, g \in G \mid gx\eta(g^{-1})x^{-1} \in Z_F \,\}.$$

If $f \in C(G, \widetilde{\omega})$ and x is η-semisimple, that is, (x, η) is a semisimple element of $G \rtimes \langle \eta \rangle$, the *$\eta$-twisted orbital integral of f at x* is defined by ([G2], Def 1.9):

$$\Phi_\eta(x, f) = \int_{G/G_{x\eta, Z_F}} f(gx\eta(g^{-1}))\, dg^\times,$$

where $dg^\times$ is the G-invariant measure on the quotient coming from Haar measures on G and $G_{x\eta, Z_F}$.

Let G', resp. G'', be the F-rational points of the quasi-split unitary group in $2n$, resp. $2n+1$, variables defined with respect to K/F. Let P', resp. P'', be a maximal parabolic subgroup of G', resp. G'', having Levi component isomorphic to G, resp. $G \times K^1$ (see [G2], §§2,6). Let π be an irreducible supercuspidal representation of G. Given a character ξ of K^1, define a supercuspidal representation Π_ξ of $G \times K^1$ by

$$\Pi_\xi(x, \alpha) = \pi(x)\, \xi(\det_0(x\eta(x))\alpha), \qquad x \in G,\ \alpha \in K^1.$$

Here $\det_0$ denotes the determinant on G. Extend π, resp. Π_ξ, trivially across the unipotent radical to obtain a representation $\pi \otimes 1$, resp. $\Pi_\xi \otimes 1$, of P', resp. P''. Set $I(\pi) = \mathrm{Ind}_{P'}^{G'}(\pi \otimes 1)$ and $I(\Pi_\xi) = \mathrm{Ind}_{P''}^{G''}(\Pi_\xi \otimes 1)$. When $n = 1$, $I(\pi)$ and $I(\Pi_\xi)$ are principal series representations, and it is known when such representations are reducible ([K1], [K2]). Thus we will assume that $n \ge 2$.

Let h_1 and h_2 be inequivalent hermitian matrices in G. Then h_1 is stably η-conjugate to h_2 ([G2], Definition 1.3, Corollary 1.7). This implies ([R]) that $G_{h_1\eta, Z_F}$ is (the F-rational points of) an inner form of $G_{h_2\eta, Z_F}$. Using an inner twisting, we define compatible measures on $G_{h_1\eta, Z_F}$ and G_{h_2,η, Z_F}, and hence on the quotients $G/G_{h_k\eta, Z_F}$, $k = 1, 2$.

Theorem 4.1. *([G2], Theorem 2.9, Theorem 6.3) Let π be an irreducible unitary supercuspidal representation of G such that $\pi \circ \eta \sim \pi$. Then the following are equivalent:*
(i) $I(\pi)$ is reducible.
(ii) $I(\Pi_\xi)$ is irreducible (for any ξ).
(iii) $\Phi_\eta(h_1, f) + \Phi_\eta(h_2, f) = 0$ for every matrix coefficient f of π.

Remark 4.2.
(1) The condition $\pi \circ \eta \sim \pi$ is necessary for reducibility of either of $I(\pi)$ and $I(\Pi_\xi)$ ([G2]).
(2) The sum $\Phi_\eta(h_1, f) + \Phi_\eta(h_2, f)$ can be expressed as a κ-twisted orbital integral of f (see §1 of [G2]).
(3) The reducibility of $I(\Pi_\xi)$ is independent of the choice of ξ ([G2], §6).

As in §2, let E be a tamely ramified degree n extension of K, and take θ to be a unitary character of $E^\times$ which is admissible over K and satisfies $\theta^{-1} = \theta \circ \sigma$ for some $\sigma \in Aut(E/F)$ such that $\sigma \,|\, K$ is non-trivial. Let π be the irreducible supercuspidal representation of G associated to θ via Howe's construction. Let H_0 be the open compact-mod-centre subgroup of G defined in §3. Then $\pi = \mathrm{Ind}_{H_0}^G \kappa$ for some irreducible representation κ of H_0. Let χ_κ denote the character of κ. Set

$$\dot{\chi}_\kappa(x) = \begin{cases} \chi_\kappa(x), & \text{if } x \in H_0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Then the function f_π defined by $f_\pi(x) = \dot{\chi}_\kappa(xh_1^{-1})$ is a finite sum of matrix coefficients of π.

Let h_k, $k = 1, 2$, and φ be as in §3. Recall that $h_2 = a_L h_1$ and $a_L \in L^\times$, where $L = E^\sigma$. Then, noting that $\varphi(g) = h_1 \eta(g^{-1}) h_1^{-1}$, we have

$$\begin{aligned} \Phi_\eta(h_1, f_\pi) &= \int_{G/G_{h_1\eta, Z_F}} \dot{\chi}_\kappa\big(g\varphi(g)\big)\, dg^\times, \\ \Phi_\eta(h_2, f_\pi) &= \int_{G/G_{h_2\eta, Z_F}} \dot{\chi}_\kappa\big(ga_L\varphi(g)\big)\, dg^\times. \end{aligned} \tag{4.2}$$

Our aim is to show that under certain conditions on θ, both of the integrals $\Phi_\eta(h_k, f_\pi)$, $k = 1, 2$, are positive and hence, by Theorem 4.1, that $I(\pi)$ is irreducible, and $I(\Pi_\xi)$ is reducible.

5. Preliminary results

Let the subgroups H_0, $\mathcal{K}_i$, $\mathcal{L}_i$, etc. be defined as in §3. For $0 \le i \le r$, let $\det_i : M_i \to E_i$ denote the determinant on $M_i \simeq \mathfrak{gl}_{[E:E_i]}(E_i)$. The notation tr will be used for the trace map on $\mathfrak{gl}_n(K)$. Recall ([H], [M]) that $\pi = \mathrm{Ind}_{H_0}^G \kappa$, where the inducing representation κ is a tensor product:

$$\kappa = (\Lambda \circ \det_0) \otimes \kappa_1 \otimes \cdots \otimes \kappa_r,$$

and κ_i is defined using the character θ_i of $E_i^\times$ which appears in the Howe factorization of θ. We continue to assume that Λ and θ_i, $1 \le i \le r$, are chosen as in Lemma 2.3. When $f_E(\theta_i \circ N_{E/E_i}) > 1$, the representation κ_i is first defined on $E^\times \mathcal{K}_{i-1}$ and then extended across $\mathcal{L}_{i-1}$ by $\psi(\mathrm{tr}(c_i(\cdot - 1)))$ to get a representation of $H_0 = E^\times \mathcal{K}_{i-1} \mathcal{L}_{i-1}$. Here, $c_i \in E_i$ is an element representing θ_i as in Lemma 2.3.

If $f_E(\theta_r) = 1$ then κ_r is defined in terms of the cuspidal representation of the finite general linear group $P(r-1)/P_1(r-1)$ parametrized by $\theta_r \,|\, \mathcal{O}_E^\times$. This case will be discussed in §7.

Recall that $m_i = \left[\frac{f_E(\theta_i \circ N_{E/E_i})+1}{2}\right]$ and $\ell_i = \left[\frac{f_E(\theta_i \circ N_{E/E_i})}{2}\right]$, $1 \leq i \leq r$. If $i \leq r-1$ or if $i = r$ and $f_E(\theta_r) > 1$, define a character ω_i of $E^\times \mathcal{K}_i P_{m_i}(i-1)\mathcal{L}_{i-1} \subset H_0$ by

$$\omega_i \,|\, E^\times \mathcal{K}_i = \theta_i \circ \det_i \qquad \text{and} \qquad \omega_i \,|\, P_{m_i}(i-1)\mathcal{L}_{i-1} = \psi(\mathrm{tr}(c_i(\cdot - 1))).$$

The condition $2m_i \geq f_E(\theta_i \circ N_{E/E_i})$ guarantees that the two definitions coincide on the intersection $E^\times \mathcal{K}_i \cap P_{m_i}(i-1)\mathcal{L}_{i-1}$ ([H]).

If $x \in H_0^\varphi$, then, by Lemma 3.2(i), $x \in L^\times P_1(0)$. For $x \in H_0^\varphi$, define

$$\mu(x) = \begin{cases} 1, & \text{if } x \in N_{E/L}(E^\times)\, P_1(0), \\ a_L, & \text{otherwise.} \end{cases}$$

Lemma 5.1. *(*[MR]*, Lemma 5.1) If E/L is ramified, then $f_E(\theta_r) > 1$.*

Lemma 5.2. *(*[MR]*, Lemma 5.2)*
(i) Suppose that $x \in E^\times \mathcal{K}_i P_{m_i}(i-1)\mathcal{L}_{i-1}$ and $\varphi(x) = x$. If $f_E(\theta_r) = 1$, make the additional assumption that $x \in E^\times P_1(0)$. Then $\omega_i(x) = \theta_i(N_{E/E_i}(\mu(x)))$.
(ii) If $x \in H_0^\varphi$, then $\Lambda(\det_0(x)) = \Lambda(N_{E/K}(\mu(x)))$.

The conductoral exponent $f_E(\theta_i \circ N_{E/E_i})$ is even if and only if $m_i = \ell_i$. In this case, $E^\times \mathcal{K}_i P_{m_i}(i-1) = E^\times \mathcal{K}_{i-1}$, so ω_i is defined on all of H_0, and $\kappa_i = \omega_i$. In particular, if $m_i = \ell_i$, then $\dim \kappa_i = 1$. If $i = r$ and $f_E(\theta_r) = 1$, since the construction of κ_r involves a cuspidal representation of a finite general group, we have $\dim \kappa_r > 1$. Otherwise, $m_i = \ell_i + 1 \geq 2$ and a Heisenberg construction is used to define κ_i on $E^\times \mathcal{K}_i$, and $\dim \kappa_i > 1$.

Proposition 5.3. *If $\dim \kappa = 1$ and $\theta \,|\, L^\times \equiv 1$, then $\Phi_\eta(h_k, f_\pi) > 0$, $k = 1, 2$.*

Proof. By the above remarks, $m_i = \ell_i$, $1 \leq i \leq r$, and $\kappa_i = \omega_i$. If $x = g\varphi(g) \in H_0$, then $\varphi(x) = x$, so Lemma 5.2 applies and

$$\kappa(g\varphi(g)) = \Lambda(\det_0(g\varphi(g))) \prod_{i=1}^{r} \kappa_i(g\varphi(g))$$

$$= \Lambda(N_{E/K}(\mu(g\varphi(g)))) \prod_{i=1}^{r} \theta_i(N_{E/E_i}(\mu(g\varphi(g)))) = \theta(\mu(g\varphi(g))), \quad \text{if } g\varphi(g) \in H_0.$$

Similarly, if $ga_L\varphi(g) \in H_0$, by Lemma 5.2,

$$\kappa(ga_L\varphi(g)) = \Lambda(N_{E/K}(\mu(ga_L\varphi(g)))) \prod_{i=1}^{r} \kappa_i(ga_L\varphi(g)) = \theta(\mu(ga_L\varphi(g))).$$

Since $\mu(x) \in L^\times$ for $x \in H_0^\varphi$ and $\theta \,|\, L^\times \equiv 1$, it follows from (4.2) that $\Phi_\eta(h_k, f_\pi) = \Phi_\eta(h_k, \mathbf{1}_{H_0 h_1})$, $k = 1, 2$, where $\mathbf{1}_{H_0 h_1}$ denotes the characteristic function of $H_0 h_1$. Since $P_j(0)$ and $a_L P_j(0)$ are contained in H_0 for sufficiently large j, it is a simple matter to show, using Lemma 3.2(ii), that $\Phi_\eta(h_k, \mathbf{1}_{H_0 h_1}) > 0$, $k = 1, 2$. □

We collect some results of [MR] which will be used later in this paper.

Lemma 5.4. *(*[MR]*, Lemmas 5.4–5.7)*
(i) Suppose that $K \subset N_1 \subset N_2 \subset E$, $\sigma(N_j) = N_j$, $j = 1, 2$, and $N_2/(N_2 \cap L)$ is ramified. Then $N_1/(N_1 \cap L)$ is ramified and $e(N_2/N_1)$ is odd.
(ii) If E/L is ramified, then $\dim \kappa = 1$.

(iii) If a Heisenberg construction is required for one of the κ_i's, then E/L is unramified.
(iv) If $r > 1$, $f_E(\theta_r) = 1$, and $e(E_{r-1}/(E_{r-1} \cap L)) = 2$, then $\dim \kappa_i = 1$ for $1 \leq i \leq r-1$.

6. The Heisenberg construction

Fix i, $1 \leq i \leq r$. Suppose that $f_E(\theta_i \circ N_{E/E_i})$ is odd, that is, $m_i = \ell_i + 1$. If $i = r$, assume in addition that $\ell_r \geq 1$. Recall that in this case (Lemma 5.4(iii)) E/L must be unramified. Set

$$H_i = K^\times(1+\mathfrak{p}_E)(\mathcal{K}_i P_{\ell_i}(i-1) \cap P_1(0)),$$
$$H_i' = K^\times(1+\mathfrak{p}_E)(\mathcal{K}_i P_{m_i}(i-1) \cap P_1(0)).$$

Let ω_i be the character of $E^\times \mathcal{K}_i P_{m_i}(i-1)\mathcal{L}_{i-1}$ defined in §5. Let χ_i denote the character of κ_i. A Heisenberg construction is used to define $\kappa_i \,|\, E^\times \mathcal{K}_i P_{\ell_i}(i-1)$ in such a way that the restriction of χ_i to H_i' is a multiple of $\omega_i \,|\, H_i'$. Then, if $i \geq 2$, κ_i is extended by $\psi(\mathrm{tr}(c_i(\cdot - 1)))$ on $\mathcal{L}_{i-1}$ to produce a representation of H_0. In this section, we see that, for $x \in (E^\times H_i)^\varphi$, $\chi_i(x)$ is a real scalar multiple of $\theta_i(N_{E/E_i}(\mu(x)))$. When the scalar multiple is non-zero, we compute its sign (Corollary 6.5).

If $F \subset N \subset E$, let ζ_N denote the set of roots of unity in N of order prime to p. We assume that a uniformizer $\varpi_N \in N$ is chosen so that $\varpi_N^{e(N/F)} \in \varpi\,\zeta_F$, where ϖ is a uniformizer in F. Let C_N be the subgroup of $N^\times$ generated by ϖ_N and ζ_N.

Lemma 6.1. *Let $x \in L^\times(H_i \cap P_1(0))$.*
(i) There exists a unique $c_L(x) \in C_L$ such that $x \in c_L(x)(H_i \cap P_1(0))$.
(ii) Suppose that $y^{-1}xy \in E^\times H_i'$ for some $y \in E^\times H_i$. Then, given any subfield N of E containing $E_0 = K$,

$$y^{-1}xy \in N^\times H_i' \Longleftrightarrow c_L(x) \in N^\times.$$

Remark 6.2. In [MR], an analogue of the above lemma was proved for points which were φ-invariant, but the proof only required $x \in L^\times(H_i \cap P_1(0))$.

Define

$$\mathcal{S}_i = \{\, N \mid E_{i-1} \subset N \subset E,\ N \not\supset E_i \,\},$$

To each $N \in \mathcal{S}_i$, there are attached a sign $\mathrm{sgn}(N) \in \{\pm 1\}$, and a positive integer $D(N)$ as defined in (3.6.47) of [M]. Set

$$\mathrm{sgn}(x) = \prod_{\{N \in \mathcal{S}_i \,\mid\, c_L(x) \notin N^\times\}} \mathrm{sgn}(N), \qquad x \in L^\times(H_i \cap P_1(0)).$$

Let $q_{E_{i-1}}$ denote the cardinality of the residue class field of E_{i-1}.

Lemma 6.3. *Let $x \in (E^\times H_i)^\varphi$. If x is conjugate to an element of $E^\times H_i'$, then*

$$\chi_i(x) = q_{E_{i-1}}^{\sum_{\{N \in \mathcal{S}_i \,\mid\, c_L(x) \in N^\times\}} D(N)}\, \mathrm{sgn}(x)\, \theta_i(N_{E/E_i}(\mu(x))).$$

Otherwise $\chi_i(x) = 0$. Here, μ is as defined in §5.

Proof. The second statement of the lemma follows from [M], §3.6. Thus, without loss of generality, we assume that there exists $y \in E^\times H_i$ such that $y^{-1}xy \in E^\times H_i'$.

Let ω_i and μ be defined as in §5. It follows from results of [M] (see Lemma 6.1 of [MR]) and Lemma 6.1, that

$$\chi_i(x) = q_{E_{i-1}}^{\sum_{\{N\in\mathcal{S}_i \mid c_L(x)\in N^\times\}} D(N)} \operatorname{sgn}(x)\,\omega_i(y^{-1}xy).$$

To complete the proof, arguing as for Lemma 6.4 of [MR] results in:

$$\omega_i(y^{-1}xy) = \theta_i(N_{E/E_i}(\mu(x))).$$

□

Lemma 6.4. *Let $L' = L_{un}(\varpi_L\sqrt{\varepsilon})$, where L_{un} is the maximal unramified extension of F contained in L and ε is a non-square in $\zeta_{L_{un}}$. Suppose that $N \in \mathcal{S}_i$ and $\sigma(N) = N$.*
(i) If K/F is unramified, then $\operatorname{sgn}(N) = 1$.
(ii) If K/F is ramified and $e(E/K)$ is even, then $\operatorname{sgn}(N) = 1$.
(iii) If K/F is ramified, $e(E/K)$ is odd, and $e(E_{i-1}/(E_{i-1}\cap L)) = e(E_i/(E_i \cap L))$, then $\operatorname{sgn}(N) = 1$.
(iv) If K/F is ramified, $e(E/K)$ is odd, $e(E_i/(E_i\cap L)) = 1$ and $e(E_{i-1}/(E_{i-1}\cap L)) = 2$, then

$$\operatorname{sgn}(N) = \begin{cases} -1, & \text{if } N = L', \\ 1, & \text{otherwise.} \end{cases}$$

Proof. As shown in Proposition 3.6.55 of [M], $\operatorname{sgn}(N) = 1$ whenever $f(E/N) > 2$. By arguing as in the second part of the proof of Lemma 7.4 of [MR], we see that $\operatorname{sgn}(N) = 1$ whenever $f(E/N) = 1$. Thus we need only consider the case $f(E/N) = 2$.

Suppose that K/F is unramified. As E/L is also unramified, and $K \not\subset L$, we have $f(E/K) = f(L/F)$ odd. In particular, as $K \subset N$, $f(E/N)$ must be odd, and so (i) follows.

Suppose that K/F is ramified. Assume that $N \in \mathcal{S}_i$, $f(E/N) = 2$ and $\sigma(N) = N$. Then, by Proposition 7.6 of [MR], $\operatorname{sgn}(N) = 1$ if $[E : N] > 2$, and $\operatorname{sgn}(N) = -1$ if $[E : N] = 2$. By Lemma 7.5(i) of [MR], L and L' are the only two extensions N' of F in E satisfying $\sigma(N') = N'$ and $[E : N'] = f(E/N') = 2$. By Lemma 7.5(ii) of [MR], $K \subset E_{i-1} \subset L'$ is equivalent to $e(E/K)$ odd and $e(E_{i-1}/(E_{i-1}\cap L)) = 2$. Also, if $e(E/K)$ is odd, then $L' \not\supset E_i$ is equivalent to $e(E_i/(E_i \cap L)) = 1$. Thus, by definition of $\mathcal{S}_i$, $L' \in \mathcal{S}_i$ is equivalent to the three conditions $e(E/K)$ odd, $e(E_{i-1}/(E_{i-1}\cap L)) = 2$ and $e(E_i/(E_i \cap L)) = 1$. Parts (ii)–(iv) now follow. □

Corollary 6.5. *Let $x \in L^\times(H_i \cap P_1(0))$. Then, if ν is as defined in §3,*

$$\operatorname{sgn}(x) = \begin{cases} (-1)^{\nu(x)}, & \text{if } e(E/K) \text{ is odd, } e(E_{i-1}/(E_{i-1}\cap L)) = 2, \\ & \quad \text{and } e(E_i/(E_i\cap L)) = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. First note that if $\sigma(N) \neq N$, then $\sigma(E_{i-1}) = E_{i-1}$ implies that $E_{i-1} \subset \sigma(N)$. Also, $E_i \not\subset N$ and $\sigma(E_i) = E_i$ implies that $E_i \not\subset \sigma(N)$. Thus if $\sigma(N) \neq N$, we have $N \in \mathcal{S}_i$ if and only if $\sigma(N) \in \mathcal{S}_i$. It follows easily from the definitions in [M] that $\operatorname{sgn}(N) = \operatorname{sgn}(\sigma(N))$. Therefore, when computing $\operatorname{sgn}(x)$, we need only consider those $N \in \mathcal{S}_i$ such that $c_L(x) \notin N^\times$ and $\sigma(N) = N$.

It now follows from Lemma 6.4 that we need only consider the case where $e(E/K)$ is odd, $e(E_{i-1}/(E_{i-1}\cap L)) = 2$, and $e(E_i/(E_i \cap L)) = 1$. (Note that in this case

K/F is ramified, by Lemma 5.4(i)). By Lemma 6.4(iv),

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } c_L(x) \notin L'^{\times}, \\ 1, & \text{if } c_L(x) \in L'^{\times}. \end{cases}$$

By definition, L' is a quadratic extension of $L_{un}(\varpi_L^2)$ containing ζ_L and not containing ϖ_L. It is immediate that $c_L(x) \in L'^{\times}$ if and only if $c_L(x) \in \varpi_L^{2k}\zeta_L$ for some integer k; that is, if and only if $\nu(c_L(x)) = \nu(x)$ is even. □

We can predict precisely when there will be a Heisenberg construction with $\operatorname{sgn}(x) = -1$ for some x, as follows:

Lemma 6.6. *Assume that E/L is unramified.*
(i) Suppose that K/F is ramified and $e(E/K)$ is odd. Then there exists a unique j, $1 \le j \le r$, having the property that $m_j = \ell_j + 1$ and $\operatorname{sgn}(x) = (-1)^{\nu(x)}$, $x \in L^{\times}(H_j \cap P_1(0))$. In particular, for all $i \ne j$, $1 \le i \le r$, such that $m_i = \ell_i + 1$, we have $\operatorname{sgn}(x) = 1$, for every $x \in L^{\times}(H_i \cap P_1(0))$.
(ii) If the conditions of (i) are not satisfied, then for all i, $1 \le i \le r$, such that $m_i = \ell_i + 1$, we have $\operatorname{sgn}(x) = 1$, for every $x \in L^{\times}(H_i \cap P_1(0))$.

Proof. First suppose that K/F is ramified and $e(E/K)$ is odd. Then, by Lemma 5.4(i), there exists a unique j, $1 \le j \le r$, such that $e(E_j/(E_j \cap L)) = 1$ and $e(E_{j-1}/(E_{j-1} \cap L)) = 2$. As $e(E/E_j)$ is odd, it follows from

$$f_E(\theta_j \circ N_{E/E_j}) = e(E/E_j)\,(f_{E_j}(\theta_j) - 1) + 1$$

that $m_j = \ell_j + 1$ if and only if $f_{E_j}(\theta_j)$ is odd. To show that $f_{E_j}(\theta_j)$ is odd, argue as in the proof of Corollary 7.11 of [MR]. All statements concerning $\operatorname{sgn}(x)$ are now immediate consequences of Corollary 6.5. □

7. The case $f_E(\theta_r) = 1$.

Throughout this section, we assume that $f_E(\theta_r) = 1$ and that if $r > 1$, then κ_j is one-dimensional for $1 \le j \le r-1$. Using a modification of the arguments of §10 of [MR], we express each $\Phi_\eta(h_k, f_\pi)$, $k = 1, 2$, in terms of sums of the character χ_r of κ_r over subsets of $\overline{H}_0$. Certain conditions on θ imply that $\Phi_\eta(h_k, f_\pi) > 0$. Omitting some of the details, we indicate how to adapt the results of §10 of [MR] to this setting.

We now define prime elements in E, L, E_{r-1} and $E_{r-1} \cap L$ as in [MR]. Recall that $f_E(\theta_r) = 1$ implies that E is unramified over L (Lemma 5.1) and over E_{r-1}. Set $e_0 = e(E_{r-1}/E_{r-1} \cap L)$ and $f_0 = f(E_{r-1}/E_{r-1} \cap L)$. Fix a prime element ϖ_0 in $E_{r-1} \cap L$ and a non-square root of unity ε in L. If $e_0 = 1$, then $E/(E_{r-1} \cap L)$ is unramified and we choose prime elements in E and L as follows: $\varpi_E = \varpi_L = \varpi_0$. If $e_0 = 2$, then $\varpi_E = \sqrt{\varpi_0}$ is a prime element in E which generates E_{r-1} over $E_{r-1} \cap L$ and satisfies $\sigma(\varpi_E) = -\varpi_E$. Furthermore, the element $\varpi_L = \sqrt{\varepsilon\varpi_0} = \sqrt{\varepsilon}\varpi_E$ is a prime element in L.

Let $\overline{M}$ denote the residue class field of a p-adic field M. Set

$$\overline{H}_0 = (H_0 \cap P(0))/(H_0 \cap P_1(0)).$$

It follows from the definition of H_0 that

$$\overline{H}_0 \simeq P(r-1)/P_1(r-1) \simeq GL_{[E:E_{r-1}]}(\overline{E}_{r-1}).$$

If $r > 1$, then $\overline{H}_0 = \overline{H}$, where $\overline{H} = (H \cap P)/(H \cap P_1)$ is as in §9 of [MR]. If $r = 1$, then since $E_0 = K$ here and the E_0 of [MR] was F, we have $\overline{H}_0 = \overline{H} \cap GL_n(\overline{K})$.

We can now apply the results of §9 of [MR], remembering to replace $\overline{H}$ by $\overline{H}_0$ in the case $r = 1$.

As $f_E(\theta_r) = 1$ and θ_r is generic over E_{r-1}, the character $\theta_r \,|\, \mathcal{O}_E^\times$ determines a character of $\overline{E}^\times$ which corresponds to an irreducible cuspidal representation $\overline{\kappa}_r$ of $\overline{H}_0$. The restriction of κ_r to $H_0 \cap P(0)$ is trivial on $H_0 \cap P_1(0)$ and induces $\bar{\kappa}_r$ on $\overline{H}_0$. As the prime element ϖ_E above is a prime element in E_{r-1}, setting $\kappa_r(\varpi_E) = \theta_r(\varpi_E)\,\kappa_r(1)$ extends κ_r to H_0.

Let $\mathcal{C}_{\overline{E}}$, resp. $\mathcal{C}_{\overline{L}}$, be the set of elements in $\overline{H}_0$ whose semisimple part is conjugate to an element of $\overline{E}$, resp. $\overline{L}$. Next, define $\mathcal{S}_{E-L}$, resp. $\mathcal{S}_L$, to be the set of $x \in H_0 \cap P(0)$ such that the image of x in $\overline{H}_0$ belongs to $\mathcal{C}_{\overline{E}} \backslash \mathcal{C}_{\overline{L}}$, resp. $\mathcal{C}_{\overline{L}}$. It follows from properties of the cuspidal representation $\overline{\kappa}_r$ of $\overline{H}_0$ that if $x \in H_0 \cap P(0)$ does not belong to $\mathcal{S}_L \cup \mathcal{S}_{E-L}$, then $\chi_r(x) = 0$. As we will see in Lemma 7.2, we need only consider values of χ_r for $x \in (\varpi_E^k(H_0 \cap P(0)))^\varphi$, $k = 1, 2$. The following lemma gives information on properties of such x, when $\varpi_E^{-k} x \in \mathcal{S}_L \cup \mathcal{S}_{E-L}$, $k = 1, 2$.

Lemma 7.1.
(i) Suppose that $x \in P(r-1)^\varphi$. Then there exists $g \in P(r-1)$ such that $x = g\varphi(g)$.
(ii) Suppose that $x \in (\varpi_E P(r-1))^\varphi$. If $e_0 = 2$ and $\varpi_E^{-1} x \in \mathcal{S}_{E-L}$, or if $e_0 = 1$, then there exists $g \in P(r-1)$ such that $x = g\varpi_L\varphi(g)$.
(iii) Suppose that $e_0 = 2$ and $f(L/(E_{r-1} \cap L))$ is even. Fix $\delta \in P(r-1) \cap \mathcal{S}_L$ such that $\varphi(\varpi_E\delta) = \varpi_E\delta$. If $x \in (\varpi_E P(r-1))^\varphi$ and $\varpi_E^{-1} x \in \mathcal{S}_L$, then there exists $g \in P(r-1)$ such that $x = g\varpi_E\delta\varphi(g)$. Furthermore, $x = g_1\varphi(g_1)$ for some $g_1 \in G$.

Proof. Statements (i), (ii), and the first part of (iii) are proved as in Lemma 10.3 of [MR].

Recall that $h_2 = \varpi_L h_1$ (§3). Given $y \in G$, $y \in G^\varphi$ if and only if yh_1 is hermitian. Recall (§3) that h_1 and h_2 belong to distinct equivalence classes of hermitian matrices. It follows that G^φ is the disjoint union of the sets $\{\, g\varpi_L^\ell \varphi(g) \mid g \in G \,\}$, $\ell = 0, 1$, the elements of the first set, resp. second set, having determinants in $N_{K/F}(K^\times)$, resp. in $N_{E/K}(\varpi_L)\, N_{K/F}(K^\times) = F^\times \backslash N_{K/F}(K^\times)$. Assume that δ is as in (iii). As $\varpi_E\delta \in G^\varphi$ by assumption, to show that $\varpi_E\delta = y\varphi(y)$ for some $y \in G$, it suffices to show that $\det_0(\varpi_E\delta) \in N_{K/F}(K^\times)$.

Because $\delta \in \mathcal{S}_L$,

$$\det{}_0(\delta) \in \det{}_0(\mathcal{O}_L^\times) = \det{}_0(N_{E/L}(\mathcal{O}_E^\times)) \subset N_{K/F}(\mathcal{O}_K^\times).$$

Also, by choice of the prime element $\varpi_E \in E_{r-1}$, since

$$[E : E_{r-1}] = 2\, f(L/(E_{r-1} \cap L))$$

is divisible by 4, $N_{E/E_{r-1}}(\varpi_E) = \det_{r-1}(\varpi_E) = \varpi_E^{[E:E_{r-1}]} \in (E_{r-1} \cap L^\times)^2$. Thus $\det_0(\varpi_E) = N_{E/K}(\varpi_E) \in (F^\times)^2$. We conclude that $\det_0(\varpi_E\delta) \in N_{K/F}(K^\times)$. Thus $\varpi_E\delta = y\varphi(y)$, for some $y \in G$. Taking x as in (iii), there exists $g \in P(r-1)$ such that $x = g\varpi_E\delta\varphi(g) = gy\varphi(gy)$. Set $g_1 = gy$. □

Let $\mathcal{F}_k = f_\pi \cdot \mathbf{1}_{(H_0 \cap P(0))h_k}$, $k = 1, 2$, where we write $\mathbf{1}_S$ for the characteristic function of a subset S of G.

Lemma 7.2. *Set $e = e(E/F)$. Let $(\cdot,\cdot)$ denote gcd.*
(i) $\Phi_\eta(h_1, f_\pi) = \frac{e}{(2,e)}\,(\Phi_\eta(h_1, \mathcal{F}_1) + \Phi_\eta(h_1, \mathcal{F}_2))$.
(ii) $\Phi_\eta(h_2, f_\pi) = \frac{e}{(2,e)}\Phi_\eta(h_2, \mathcal{F}_2)$.

Proof. By arguing as in the proof of Lemma 10.1 of [MR],

$$\chi_\kappa(\varpi_E^j x\varphi(\varpi_E^j)) = \chi_\kappa(x), \qquad x \in H_0.$$

Let $C_k = \{\, g\varpi_L^{k-1}\varphi(g) \mid g \in G\,\}$, $k = 1, 2$. Recall (see above) that G^φ is the disjoint union of C_1 and C_2. Given $j \in \mathbb{Z}$ and $\alpha \in \mathcal{O}_E^\times$, define a map $\lambda_{\alpha,j}$ from G to G by $\lambda_{\alpha,j}(x) = \varpi_E^j\alpha x\varphi(\varpi_E^j\alpha)$. For $1 \le k, \ell \le 2$, the map $\lambda_{\alpha,j}$ restricts to a measure-preserving bijection between

$$C_k \cap \varpi_E^{\ell-1}(H_0 \cap P(0)) \quad \text{and} \quad C_k \cap \varpi_E^{\ell-1+2j}(H_0 \cap P(0)),$$

where the measure is the one on $G/G_{h_k\eta,Z_F}$. Thus, using the map $\lambda_{\alpha,j}$, and the fact that $\chi_\kappa \circ \lambda_{\alpha,j} = \chi_\kappa \circ \lambda_{\alpha,0}$ (see above),

$$\begin{aligned}
&\int_{G/G_{h_k\eta,Z_F}} (\dot\chi_\kappa \mathbf{1}_{\varpi_E^{\ell-1}(H_0\cap P(0))})(g\varpi_L^{k-1}\varphi(g))\, dg^\times \\
&= \int_{G/G_{h_k\eta,Z_F}} \dot\chi_\kappa(\alpha g\varpi_L^{k-1}\varphi(\alpha g))\, \mathbf{1}_{\varpi_E^{\ell-1+2j}(H_0\cap P(0))}(g\varpi_L^{k-1}\varphi(g))\, dg^\times \qquad (7.1)\\
&= \int_{G/G_{h_k\eta,Z_F}} (\dot\chi_\kappa \mathbf{1}_{\varpi_E^{\ell-1+2j}(H_0\cap P(0))})(g\varpi_L^{k-1}\varphi(g))\, dg^\times.
\end{aligned}$$

To obtain the second equality, we have used the fact that $\lambda_{\alpha,0}$ fixes the set $\varpi_E^{\ell-1+2j}(H_0 \cap P(0))$.

The smallest positive integer j such that $N_{E/L}(\varpi_E^j\mathcal{O}_E^\times) \cap F^\times \neq \emptyset$, that is, such that $\varpi_E^j\mathcal{O}_E^\times \cap G_{h_k\eta,Z_F} \neq \emptyset$, is $j = e/(2,e)$. Therefore, applying (7.1) (which is independent of the choice of $\alpha \in \mathcal{O}_E^\times$), we conclude from (4.1) and $H_0 = \bigcup_{j\in\mathbb{Z}} \varpi_E^j(H_0 \cap P(0))$, that

$$\Phi_\eta(h_k, f_\pi) = \frac{e}{(2,e)} \sum_{1\le \ell\le 2} \int_{G/G_{h_k\eta,Z_F}} \left(\dot\chi_\kappa\, \mathbf{1}_{\varpi_E^{\ell-1}(H_0\cap P(0))}\right)(g\varpi_L^{k-1}\varphi(g))\, dg^\times, \qquad k = 1, 2.$$

As $h_2 = \varpi_L h_1$, ϖ_L normalizes $H_0 \cap P(0)$, and $\varpi_E \in \varpi_L\mathcal{O}_E^\times \subset \varpi_L(H_0 \cap P(0))$, it follows that $(H_0 \cap P(0))h_2 = \varpi_E(H_0 \cap P(0))h_1$. Therefore (see comments preceding (4.1))

$$\Phi_\eta(h_k, \mathcal{F}_\ell) = \int_{G/G_{h_k\eta,Z_F}} \left(\dot\chi_\kappa\, \mathbf{1}_{\varpi_E^{\ell-1}(H_0\cap P(0))}\right)(g\varpi_L^{k-1}\varphi(g))\, dg^\times, \qquad k, \ell = 1, 2. \tag{7.2}$$

Comparing this with the above expression for $\Phi_\eta(h_k, f_\pi)$, we see that it remains to show that $\Phi_\eta(h_2, \mathcal{F}_1) = 0$.

Let $x \in (H_0 \cap P(0))^\varphi$. By Lemma 3.2(i), there exists $y \in (E^\times\mathcal{K}_{r-1})^\varphi = (E^\times P(r-1))^\varphi$ and $z \in \mathcal{L}_{r-1}$ such that $x = yz$. As $x \in P(0)$ and $z \in P_1(0)$, it follows that $y \in P(r-1)^\varphi$. By Lemma 7.1(i), there exists $y_1 \in P(r-1)$ such that $y = y_1\varphi(y_1)$. Since $z \in P_1(0)$, and $x = yz$ and y are φ-invariant, it follows that $\det_0(z) \in 1 + \mathfrak{p}_F$. Thus $\det_0(x) \in N_{K/F}(\det_0(y_1))(1 + \mathfrak{p}_F) \subset N_{K/F}(K^\times)$. Since $x \in C_1 \cup C_2$, $\det_0(C_1) \subset N_{K/F}(K^\times)$, and $\det_0(C_2) \subset F^\times\backslash N_{K/F}(K^\times)$, we must have $x \in C_1$. It follows from $(H_0 \cap P(0))^\varphi \subset C_1$, $C_1 \cap C_2 = \emptyset$, and (7.2) that $\Phi_\eta(h_2, \mathcal{F}_1) = 0$. □

As the κ_i's, $1 \leq i \leq r-1$, are one-dimensional, their values on the relevant φ-invariant elements in $\varpi_E^j(H_0 \cap P(0))$, $j = 1, 2$, are easily computed in terms of the characters θ_i. In [MR], this was done in Lemma 10.2. Here, the result still holds, and it is proved the same way (with $\overline{H}_0$ replacing $\overline{H}$). The computation for Λ is handled in exactly the same way. Combining this with the definition of κ_r we get, for $x \in (H_0 \cap P(0))^\varphi \cup (\varpi_E(H_0 \cap P(0)))^\varphi$,

$$\chi_\kappa(x) = \begin{cases} \theta(\varpi_E^{\nu(x)})\,\chi_r(\varpi_E^{-\nu(x)}x), & \text{if } \varpi_E^{-\nu(x)}x \in \mathcal{S}_L, \\ \theta(\varpi_L)\,\theta_r(\sqrt{\varepsilon})^{-1}\,\chi_r(\varpi_E^{-1}x), & \text{if } \nu(x) = 1 \text{ and } \varpi_E^{-1}x \in \mathcal{S}_{E-L}, \\ 0, & \text{if } \varpi_E^{\nu(x)}x \notin \mathcal{S}_L \cup \mathcal{S}_{E-L}. \end{cases} \tag{7.3}$$

Let $c = (-1)^{f_0}$. For $x \in (H_0 \cap P(0))^\varphi \cup (\varpi_E(H_0 \cap P(0)))^\varphi$, observe that the image of $\varpi_E^{-\nu(x)}x$ in $\overline{H}_0$ belongs to $\overline{H}_0^\varphi$ if $\nu(x) = 0$, and to $\overline{H}_0^{c\varphi}$ if $\nu(x) = 1$ (see the proof of Proposition 10.5 of [MR]). Here, we are using the same notation for φ and the map which φ induces on $\overline{H}_0$. The next step is to express the integrals $\Phi_\eta(h_k, \mathcal{F}_\ell)$, $1 \leq k, \ell \leq 2$, in terms of sums of χ_r over certain φ or $c\varphi$-invariant subsets of $\overline{H}_0$. This is the analogue of Proposition 10.5 of [MR]. In order to do this, we use Lemma 7.1 to write elements of $(\varpi_E^{j-1}(H_0 \cap P(r-1)))^\varphi$ in the form $g\tau\varphi(g)$, where $g \in P(r-1)$, $\tau = 1$ if $j = 1$, and $\tau \in \{\varpi_L, \varpi_E\delta\}$ if $j = 2$ (with δ as in Lemma 7.1). Using these results together with (7.2), (7.3) and Lemma 3.2, and following the proof of Proposition 10.5 of [MR], except with $\mathcal{I}(\mathcal{F}_0)$, $\mathcal{I}(\mathcal{F}_1)$, $H \cap P_1$, and $\overline{H}$ of [MR] replaced by $\Phi_\eta(h_1, \mathcal{F}_1)$, $\Phi_\eta(h_1, \mathcal{F}_2) + \Phi_\eta(h_2, \mathcal{F}_2)$, $H_0 \cap P_1(0)$, and $\overline{H}_0$, respectively, results in

Proposition 7.3. *Suppose that $f_E(\theta_r) = 1$. If $f_0 = 2$, assume that $\dim \kappa_i = 1$ for $1 \leq i \leq r-1$.*
(i) $\Phi_\eta(h_1, \mathcal{F}_1) = \Phi_\eta(h_1, \mathbf{1}_{(H_0 \cap P_1(0))h_1}) \left(\sum_{x \in \overline{H}_0^\varphi} \chi_r(x)\right)$.
(ii) If $e_0 = 1$, then $\Phi_\eta(h_1, \mathcal{F}_2) = 0$ and

$$\Phi_\eta(h_2, \mathcal{F}_2) = \theta(\varpi_L)\,\Phi_\eta(h_2, \mathbf{1}_{(H_0 \cap P_1(0))h_2}) \left(\sum_{x \in \overline{H}_0^\varphi} \chi_r(x)\right).$$

(iii) If $e_0 = 2$ and $f(L/(E_{r-1} \cap L))$ is odd, then $\Phi_\eta, (h_1, \mathcal{F}_2) = 0$ and

$$\Phi_\eta(h_2, \mathcal{F}_2) = \Phi_\eta(h_2, \mathbf{1}_{(H_0 \cap P_1(0))h_2})\,\theta(\varpi_L)\,\theta_r(\sqrt{\varepsilon})^{-1} \left(\sum_{x \in \overline{H}_0^{-\varphi}} \chi_r(x)\right).$$

(iv) If $e_0 = 2$ and $f(L/(E_{r-1} \cap L))$ is even, let δ be as in Lemma 7.1(iii). Then

$$\Phi_\eta(h_2, \mathcal{F}_2) = \Phi_\eta(h_2, \mathbf{1}_{(H_0 \cap P_1(0))h_2})\,\theta(\varpi_L)\,\theta_r(\sqrt{\varepsilon})^{-1} \left(\sum_{x \in (\mathcal{C}_{\overline{E}} \setminus \mathcal{C}_{\overline{L}}) \cap \overline{H}_0^{-\varphi}} \chi_r(x)\right),$$

$$\Phi_\eta(h_1, \mathcal{F}_2) = \Phi_\eta(h_1, \mathbf{1}_{(H_0 \cap P_1(0))\delta\sqrt{\varepsilon}^{-1}h_2})\,\theta(\varpi_E) \left(\sum_{x \in \mathcal{C}_{\overline{L}} \cap \overline{H}_0^{-\varphi}} \chi_r(x)\right).$$

Remark 7.4. We have used the facts that $(H_0 \cap P_1(0))h_2 = \varpi_L(H_0 \cap P_1(0))h_1$ and that, when $e_0 = 2$, $(H_0 \cap P_1(0))\delta\sqrt{\varepsilon}^{-1}h_2 = \varpi_E\delta(H_0 \cap P_1(0))h_1$. By arguing along the same lines as in the last part of the proof of Lemma 7.2, we can use Lemma 7.1 to show that if $x \in (\varpi_E(H_0 \cap P(0)))^\varphi$ satisfies $\varpi_E^{-1}x \in \mathcal{S}_{E-L} \cup \mathcal{S}_L$, then $x = g\varphi(g)$

for some $g \in G$ if and only if $\varpi_E^{-1} x \in \mathcal{S}_L$, and that can happen only when $e_0 = 2$ and $f(L/(E_{r-1} \cap L))$ is even. This leads to the conditions on $\Phi_\eta(h_1, \mathcal{F}_2)$ in parts (ii)-(iv) (see (7.2)).

The signs of the sums appearing in Proposition 7.3 are evaluated as in [MR], using results of §9 of [MR], yielding

$$\Phi_\eta(h_1, \mathcal{F}_1) > 0 \quad \text{and} \quad \Phi_\eta(h_1, \mathcal{F}_2) = 0,$$
$$(-1)^{f_0}\, \theta(\varpi_L)^{-1}\, \Phi_\eta(h_2, \mathcal{F}_2) > 0.$$

Combining this with Lemma 7.2 results in:

Theorem 7.5. *Suppose that $f_E(\theta_r) = 1$. If $f_0 = 2$, assume that $m_i = \ell_i$ for $1 \le i \le r-1$.*
(i) If $e_0 = 1$ and $\theta \,|\, L^\times \equiv 1$, then $\Phi_\eta(h_k, f_\pi) > 0$, $k = 1, 2$.
(ii) If $e_0 = 2$ and $\theta \,|\, L^\times \not\equiv 1$, then $\Phi_\eta(h_k, f_\pi) > 0$, $k = 1, 2$.

8. Main results

Recall that E is a tamely ramified degree $2n$ extension of F, $n \ge 2$, and θ is a unitary character of $E^\times$, admissible over the quadratic extension K of F, having the property that $\theta \circ \sigma = \theta^{-1}$ for some involution σ in $Aut(E/F)$ whose restriction to K is non-trivial. As discussed in §2 (Lemma 2.1), the supercuspidal representation π of $G = GL_n(K)$ associated to θ via Howe's construction ([H]) has the property that $\pi \circ \eta \sim \pi$. The fixed field of σ is denoted by L. Our main results are stated in terms of the values of θ on $L^\times$ and certain ramification degrees. We continue to assume that the residue characteristic p of F is odd.

Theorem 8.1. *Let f_π be the finite sum of matrix coefficients of π defined in §4. If θ satisfies one of the following conditions, then $\Phi_\eta(h_k, f_\pi) > 0$, $k = 1, 2$.*
(i) E is ramified over L and $\theta \,|\, L^\times \equiv 1$,
(ii) E is unramified over L and

$$\theta \,|\, L^\times = (-1)^{\mathrm{ord}_E(\cdot)\left(e(K/F)-1\right)e(E/K)},$$

with the additional assumption that if $r > 1$ and $f_E(\theta_r) = 1$, then $m_i = \ell_i$, $1 \le i \le r-1$.

Remark 8.2. The purpose of the additional assumption in (ii) is to exclude the case where a Heisenberg construction and a representation of a finite general linear group both occur in the inducing data for π. As remarked in [MR], we expect that the result still holds in that case.

Proof of Theorem 8.1. If (i) holds, the result follows from Proposition 5.3 and Lemma 5.4(ii).

Assume that (ii) holds. If $f_E(\theta_r) = 1$ and $f_0 = 1$, then $e(K/F) = 2$ and $e(E_{r-1}/K)$ is odd, by Lemma 5.4(i). Therefore $e(E/K) = e(E_{r-1}/K)$ is odd. Note that in this case $m_i = \ell_i$ is guaranteed by Lemma 5.4(iv). If $f_E(\theta_r) = 1$, $f_0 = 2$, and $e(K/F) = 2$, then, by Lemma 6.6(i), the assumption $m_i = \ell_i$, $1 \le i \le r-1$, implies that $e(E/K)$ is even. We conclude that in the case $f_E(\theta_r) = 1$, Theorem 7.5 coincides with this theorem.

For the remainder of the proof, suppose that (ii) holds and $f_E(\theta_r) > 1$. Let μ be as defined in §5. It follows from Lemma 5.2 that, given $1 \le i \le r$ and $x \in H_0^\varphi$,

$$\text{If } m_i = \ell_i, \text{ then } \quad \chi_i(x) = \kappa_i(x) = \theta_i(N_{E/E_i}(\mu(x))), \quad \Lambda(\det_0(x)) = \Lambda(N_{E/K}(\mu(x))). \tag{8.1}$$

Next, suppose that $m_i \ne \ell_i$ for some i. Let H_i and H_i' be as in §6. By Lemma 3.2(i), given $x \in H_0^\varphi$, there exist $y \in (E^\times H_i)^\varphi = (E^\times \mathcal{K}_{i-1})^\varphi$ and $z \in \mathcal{L}_{i-1}$ such that $x = yz$. By definition of κ_i (see the beginning of §5),

$$\chi_i(x) = \chi_i(y)\,\psi(\mathrm{tr}(c_i(z-1))).$$

Note that $z - 1 \in \mathcal{B}_{\ell_{i-1}}(0)$, so that

$$y'(z-1)y'^{-1} \in (z-1) + \mathcal{B}_{\ell_i+\ell_{i-1}}(0) \subset (z-1) + \mathcal{B}_{f_E(\theta_i \circ N_{E/E_i})}(0), \qquad \text{if } y' \in P_{\ell_i}(0).$$

Now $y \in E^\times \mathcal{K}_{i-1} = E^\times \mathcal{K}_i P_{\ell_i}(i-1)$, and c_i commutes with $E^\times \mathcal{K}_i$, so

$$\mathrm{tr}(c_i(y(z-1)y^{-1})) = \mathrm{tr}(c_i(z-1)).$$

By definition of φ, $\mathrm{tr}(\varphi(X)) = \sigma(\mathrm{tr}(X))$, $X \in \mathfrak{g}$. As x and y are φ-invariant, it follows that $\varphi(z) = yzy^{-1}$. Thus, using Lemma 2.3(iii), we find

$$\sigma(\mathrm{tr}(c_i(z-1))) = -\mathrm{tr}(c_i(y(z-1)y^{-1})) = -\mathrm{tr}(c_i(z-1)).$$

Combining this with $\psi = \psi_0 \circ \mathrm{tr}_{K/F}$ (see §2), results in $\psi(\mathrm{tr}(c_i(z-1))) = 1$. Thus $\chi_i(x) = \chi_i(y)$. As $y \in E^\times H_i$, we may apply results of §6 to evaluate $\chi_i(y)$. Let ν be as in §3. Note that $\nu(x) = \nu(y)$ and $\mu(x) = \mu(y)$. If y is conjugate to an element of $E^\times H_i'$, then by Lemma 6.3 and Corollary 6.5,

$\chi_i(x)$ is a positive multiple of

$$\begin{cases} (-1)^{\nu(x)}\,\theta_i(N_{E/E_i}(\mu(x))), & \text{if } e(E/K) \text{ is odd, } e(E_{i-1}/(E_{i-1}\cap L)) = 2, \\ & \quad \text{and } e(E_i/(E_i \cap L)) = 1, \\ \theta_i(N_{E/E_i}(\mu(x))), & \text{otherwise.} \end{cases} \tag{8.2}$$

As shown in Lemma 6.6, the first case in (8.2) can occur if and only if $e(E/K)$ is odd and $e(K/F) = 2$, and then it must occur for exactly one i, $1 \le i \le r$.

It follows from (8.1), (8.2), Lemma 6.6, and the definition of κ (see §5), that if $x \in H_0^\varphi$ and $\chi_\kappa(x) \ne 0$, then $\chi_\kappa(x)$ is a positive multiple of

$$\theta(\mu(x))\,(-1)^{\nu(x)\,(e(K/F)-1)\,e(E/K)}.$$

In particular, if $x \in (E^\times P_{m_r}(r-1)\cdots P_{m_1}(0))^\varphi$, and θ is as in (ii), then $\chi_\kappa(x) > 0$. Thus, by (4.2), $\Phi_\eta(h_k, f_\pi) > 0$, $k = 1, 2$. □

As in §4, we let G' and G'' be the F-rational points of the quasi-split unitary groups in $2n$ and $2n+1$ variables, respectively, defined with respect to K/F. Recall that P' and P'' are parabolic subgroups of G' and G'', respectively, having Levi components isomorphic to G and $G \times K^1$, respectively. Given a character ξ of K^1, the supercuspidal representation Π_ξ of $G \times K^1$ is defined by $\Pi_\xi(x,\alpha) = \pi(x)\,\xi(\det_0(x\eta(x))\alpha)$, $x \in G$, $\alpha \in K^1$. We can combine Theorem 8.1 and Goldberg's reducibility criterion (Theorem 4.1) to obtain results concerning reducibility of the representations $I(\pi) = \mathrm{Ind}_{P'}^{G'}(\pi \otimes 1)$, and $I(\Pi_\xi) = \mathrm{Ind}_{P''}^{G''}(\Pi_\xi \otimes 1)$.

Theorem 8.3. *Suppose that the admissible character θ satisfies (i) or (ii) of Theorem 8.1. Then $I(\pi)$ is irreducible and $I(\Pi_\xi)$ is reducible (for any ξ).*

It is likely that the above conditions on θ are necessary and sufficient for irreducibility of $I(\pi)$ (equivalently, for reducibility of $I(\Pi_\xi)$). See §11 of [MR] for a discussion of the analogous situation for induced representations of split classical groups. In order to show sufficiency, it would be necessary to prove that $\Phi_\eta(h_1, f) = -\Phi_\eta(h_2, f)$ for all choices of matrix coefficients f of π.

Conjecture 8.4. *$I(\pi)$ is irreducible if and only if $\theta \mid L^\times$ satisfies*

$$\theta \mid L^\times = \begin{cases} 1, & \text{if } f(E/L) = 1, \\ (-1)^{\operatorname{ord}_E(\cdot)\left(e(K/F)-1\right)e(E/K)}, & \text{if } f(E/L) = 2. \end{cases}$$

Combining Theorem 8.3 and a result of Goldberg, we can get information about reducibility of representations induced from non-unitary supercuspidal representations of G and of $G \times K^1$. Let $|\cdot|_K$ denote the normalized valuation on K. For s a non-negative real number, set

$$I(s, \pi) = I(\pi \otimes |\det_0(\cdot)|_K^{s/2})$$

and

$$I(s, \Pi_\xi) = I(\Pi_\xi \otimes |\det_0(\cdot)|_K^{s}).$$

Corollary 8.5. *Suppose that the admissible character θ satisfies (i) or (ii) of Theorem 8.1. Then $I(s, \pi)$ is reducible if and only if $s = 1$, and $I(s, \Pi_\xi)$ is irreducible for all $s > 0$.*

Proof. By Theorem 8.3, $I(\pi) = I(0, \pi)$ is irreducible, and $I(\Pi_\xi) = I(0, \Pi_\xi)$ is reducible. The result then follows from Theorems 3.1 and 6.3 of [G2]. □

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